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RESEARCH REPORT No. EM-19

EFFECT OF A CIRCULAR GROUNDPLANE ON ANTENNA RADIATION

By

ALFRED LEITNER

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APRIL, 1950

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New York University
Washington Square College
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by

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Title page
36 numbered pages
12 graphs

April, 1950

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ABSTRACT

The field of a quarter-wave dipole antenna above a circular conducting disk of zero thickness and radius a - a finite groundplane - is calculated theoretically by use of the wave functions of the oblate spheroid. The current on the groundplane, the radiation resistance and the radiation pattern of the system for groundplanes of various radii are computed. With these results the distortion of antenna radiation by groundplanes is studied. Good agreement with recent experimental data is found.

1. Introduction

Conducting objects in the neighborhood of an antenna will affect the directional distribution of its radiation because the currents which the antenna induces upon them will produce their own radiation fields. These latter fields, furthermore, will induce secondary currents on the antenna proper, with the result that the total amount $\mathcal{P}$ of power radiated by the antenna will be affected, due to the presence of such obstacles.

We shall assume that all the power dissipated in the antenna goes into its radiation. Let I_0 be the current in the circuit which supplies power to the antenna, then

$$\mathcal{P} = \frac{1}{2} \mathcal{R} I_0^2 \tag{1}$$

where $\mathcal{R}$ is the resistance of the antenna as a circuit element, as well as the measure of the power radiated: The radiation resistance.

Consequently both the radiation pattern and the circuit properties of an antenna can be affected by the presence of conductors in the radiation field.

Such conductors may be the earth itself, or metallic structures near the antenna. In a laboratory where the properties of antennas are tested, it is especially hard to isolate antennas from such objects, but also imperative to do so. For only if the properties of the completely isolated antenna are known is it possible, in an application where the antenna is not isolated, to determine the effect of the interfering secondary radiations.

Most commonly one uses groundplanes to isolate antennas. It is usually possible to mount an antenna such that all other conductors are to one side of it, say below. Then a grounded plane conducting surface is placed just below the antenna*, which is fed through a small hole in the center of the ground plane. All possible effects of this aperture in the ground plane will be neglected in this paper.

Consider first a perfectly conducting but infinite ground plane. An antenna, carrying a known current distribution on its surface, protrudes through it into empty space. It is not difficult to show that the radiation

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* often an array of grounded wires strung radially away from the antenna's axis takes the place of the above arrangement.

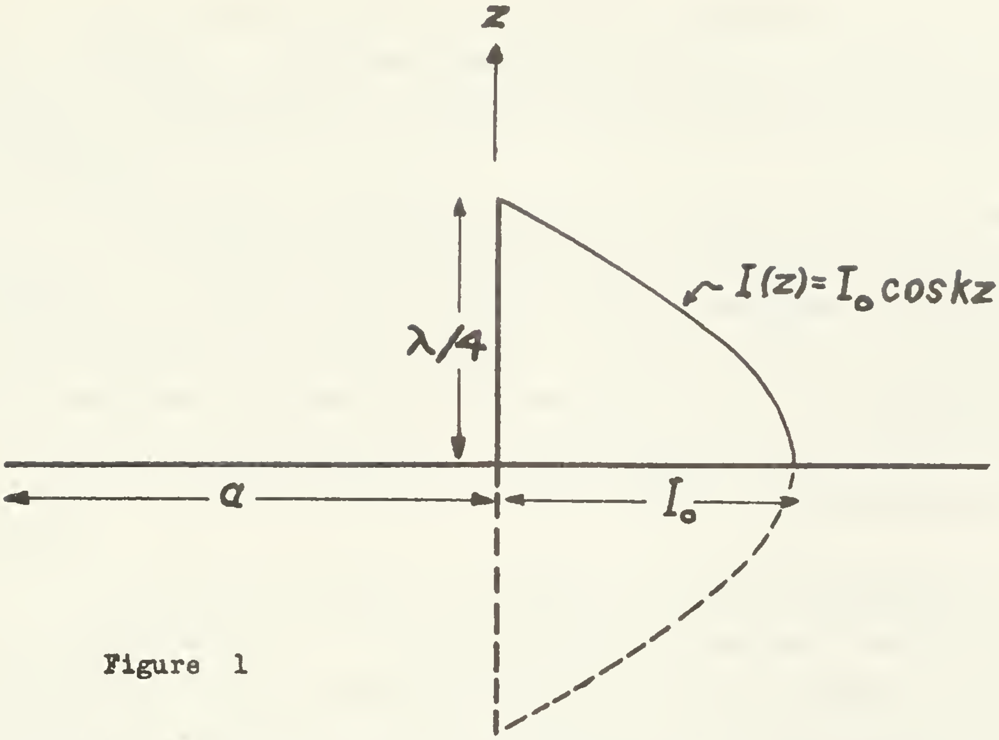

Figure 1

Quarter-wave dipole above groundplane of radius a. Dipole current I(z) plotted horizontally against corresponding value of z in solid line. Broken lines indicate mirror image of dipole and current about z = 0, which exactly describes case of infinite groundplane.

---

from this system will be as if the infinite groundplane were absent, and as if the radiation came from an antenna consisting of the following parts: (i) the original one, carrying the original current distribution, and (ii) its mirror image, carrying a mirror image of the original current distribution. For example, say that the original antenna is a quarter wave dipole on the z axis between the points $z = 0$ and $z = \frac{\lambda}{4}$, and that the groundplane lies in the xy plane. The current distribution on a dipole is sinusoidal, and in this example we take

$$(2) \qquad I(z) = I_0 \cos kz^{*} \qquad \left(k = \frac{2\pi}{\lambda}\right).$$

The field of this system above the xy plane will be exactly the same as the field of a halfwave-dipole. This halfwave dipole extends from $z = -\frac{\lambda}{4}$ to $z = \frac{\lambda}{4}$ and carries the current (2) between these points.

---

* The time dependence is harmonic, $e^{-i\omega t}$; it is implied but omitted in all expressions for electromagnetic quantities used in this work.

In a sense therefore, an infinite groundplane has no effect on antenna radiation. It is understood that the antenna to be used is really only half of the antenna to be studied. The division is made about a plane of symmetry.

In practice a groundplane must be finite. If it is large when compared to λ, it can be said to be effectively infinite. It is often impracticable to construct effectively infinite groundplanes, and groundplanes have in fact been used without regard to relative size. But then the antenna is not completely shielded from the conductors below, the groundplane itself will not act as a mirror since it is not infinite, i.e., it will itself play the role of a conducting obstacle. These effects have been neglected in practice.

In the present paper the effect of a finite groundplane is discussed by theoretical methods. An experimental investigation of the influence of groundplanes on antenna impedance was very recently published by Meier & Summers[1] independently of this work. These experiments were performed with square and circular groundplanes. The effects of these two different types on impedance were found to be qualitatively the same.

We consider circular groundplanes for mathematical convenience. In the oblate spheroidal coordinate system η, ξ, ϕ, discussed in a previous report[2], a circular disk of zero thickness is simply the surface $\xi = 0$. Maxwell's equations are separable in these coordinates, and it is possible to satisfy the boundary condition on the circular groundplane exactly by representing the field in the wave functions of the oblate spheroid.

An exact solution provides expressions for the vector field at any point of space. Thus our results are not limited to the radiation resistance of the system, which is then to be compared to the values for an infinite groundplane and to experiments. We also calculate the field on the groundplane - the surface current distribution - and the radiation pattern at large distances, quantities which have not been measured because to measure them is very difficult if not impossible. It is of especial interest to observe the effect of a finite groundplane on the radiation pattern and to compare these results with the infinite groundplane case, which produces half the pattern (at all points $z \geq 0$) of the isolated antenna.

In a theoretical paper by Bardeen[3] a problem similar to the present one is discussed, namely, the diffraction of circularly symmetric fields by a circular disk so placed that its axis coincides with the axis of symmetry. Maxwell's equations are transformed into integral equations and their solution is given in the form of integrals and does not involve the eigen functions of the disk. The intention was to evaluate these integrals for the case of a circular groundplane and any linear current distribution on its axis, but these results were not published.

2. The Antenna in Free Space

The antenna in our problem is the quarter-wave dipole introduced in the previous section, in an array with the groundplane as illustrated by figure 1. The field of this system is then compared to the field of the halfwave dipole.

A dipole is simply a sinusoidal current distribution along a straight line segment. It is, however, a good approximation to a real antenna, namely, the thin cylindrical wire with good conductivity.* There exists no exact and complete theoretical solution to the problem of the electrical oscillations and radiation of even this simplest type of antenna, and the exact solution of this problem does not concern us here[4]. Suffice it to say that the halfwave dipole of length $\frac{\lambda}{2}$ with current nodes at its ends has been shown to represent closely the lowest mode of oscillation of the perfectly conducting wire of zero thickness, as regards both the surface current distribution and the radiation field.

We shall, however, not impose any boundary conditions on the total field of the system at the points occupied by the dipole. Rather, the dipole is considered simply as a line source of radiation, and the perfectly conducting groundplane as a scatterer.

The field of the isolated dipole is discussed in the literature[4]. We confine ourselves here to a discussion of results.

If the vector separation is performed in the circular cylinder system of coordinates ρ, ϕ, z the dipole field vectors resolve into components E_z, E_ρ, and H_ϕ, which possess circular symmetry (independence of ϕ).

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*Since this is a study of groundplane effects, the shape of the antenna is of secondary importance and therefore this simplest type is chosen.

2a. The Halfwave Dipole[5]

$$(3a)\qquad E_z^{\infty} = \frac{iI_o}{4\pi}\sqrt{\frac{\mu_o}{\epsilon_o}}\left(\frac{e^{ikR_2}}{R_2} + \frac{e^{ikR_1}}{R_1}\right),$$

$$(3b)\qquad E_\rho^{\infty} = \frac{iI_o}{4\pi}\sqrt{\frac{\mu_o}{\epsilon_o}}\left[\frac{(z-\frac{\lambda}{4})}{\rho}\,\frac{e^{ikR_2}}{R_2} + \frac{(z+\frac{\lambda}{4})}{\rho}\,\frac{e^{ikR_1}}{R_1}\right],$$

$$(3c)\qquad H_\phi^{\infty} = -\frac{iI_o}{4\pi}\,\frac{1}{\rho}\,(e^{ikR_2} + e^{ikR_1}).$$

Practical units are used and μ_o and ϵ_o are the permeability and permittivity of empty space. The distance from the observation point to $z_1 = -\frac{\lambda}{4}$ is $R_1 = \sqrt{\rho^2 + (z+\frac{\lambda}{4})^2}$, to $z_2 = \frac{\lambda}{4}$ is $R_2 = \sqrt{\rho^2 + (z-\frac{\lambda}{4})^2}$.

Expressions(3)represent the field of a quarter-wave dipole above an infinite groundplane.when $z \geq 0$, as noted before. For this reason we have placed a superscript ∞ on these quantities.

The surface current density $\underset{\sim}{K}^{\infty}$ on the groundplane,

$$(4)\qquad \underset{\sim}{K}^{\infty} = \underset{\sim}{n} \times \underset{\sim}{H}^{\infty}(z=0),$$

where $\underset{\sim}{n}$ is the unit normal to the groundplane; $\underset{\sim}{K}^{\infty}$ is entirely radial and

$$(5)\qquad \begin{aligned} K_\rho^{\infty} &= -H_\phi(z=0), \\ &= \frac{I_o}{2\pi}\,\frac{ie^{ik\sqrt{\rho^2+(\lambda/4)^2}}}{\rho}, \end{aligned}$$

that is, it has a singularity at the point $\rho = 0$ just below the base of the antenna. At this latter point a current I_o flows upward on the antenna, according to (2). The total groundplane surface current

$$(6)\qquad I_\rho^{\infty}(\rho) = 2\pi\rho K_\rho^{\infty},$$

will be finite at the origin, and has the value

$$(7)\qquad I_\rho^{\infty}(0) = I_o\, ie^{ik\lambda/4} = -I_o$$

a current which flows into the origin and is in phase with and equal to the antenna current flowing away from the base.

At large distance the Poynting flux is found to be[6]

$$(8)\qquad S^{\infty} = \frac{I_0^2}{8\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{r^2}\,\frac{\cos^2\left[(\pi/2)\cos\theta\right]}{\sin^2\theta}\,,\quad r\to\infty\,,$$

where r, θ are the polar coordinates. A polar graph of the θ dependence of this function gives the radiation pattern, and is drawn in figure 7.

To calculate the total radiated power $\mathcal{P}^{\infty}$ for the infinite groundplane with the quarter-wave dipole, S^{∞} is integrated over the great hemisphere such that $\theta \leq \frac{\pi}{2}$, and $\mathcal{P}^{\infty}$ will be exactly half the power radiated from an isolated half-wave dipole. Evaluation of this integral is found in the texts[6].

$$(9)\qquad \mathcal{P}^{\infty} = 18.32\, I_0^2 \text{ watts.}$$

By use of (1), the radiation resistance of the infinite groundplane system is found to be

$$(10)\qquad \mathcal{R}^{\infty} = 36.64 \text{ ohms,}$$

half the resistance of the half-wave dipole. This value is the standard of comparision for the radiation resistance of finite groundplanes with various radii.

2b. The Quarter-wave Dipole

In this report we actually look for the total field of the radiating system, consisting of the quarter-wave dipole and the groundplane. But in the course of the solution it will be necessary at one point to consider the field of the source and the scattered field separately. We assign the superscripts d and g to them, respectively. The dipole field has the following values[7]

$$(11a)\qquad E_z^d = \frac{I_0}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left[i\,\frac{e^{ikR_2}}{R_2} - \frac{z\,e^{ikr}}{r^2}\left(1 + \frac{i}{kr}\right)\right],$$

$$(11b)\qquad E_\rho^d = -\frac{I_0}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left[i\,\frac{(z-\frac{\lambda}{4})e^{ikR_2}}{\rho R_2} - \frac{e^{ikr}}{\rho r^2}\left(z^2 + \frac{\rho^2}{ikr}\right)\right],$$

$$(11c)\qquad H_\phi^d = -\frac{I_0}{4\pi}\left(i\,\frac{e^{ikR_2}}{\rho} - \frac{z\,e^{ikr}}{\rho r}\right).$$

It is of interest to discuss at this point the detailed behavior of H^d_ϕ on the x,y plane

(12) $$H^d_\phi\,(z=0) = -\frac{I_0}{4\pi}\,\frac{ie^{ik\sqrt{\rho^2+(\lambda/4)^2}}}{\rho}\ .$$

This is one half of $H^\infty_\phi(z = 0)$. Now in a system of groundplane and quarter-wave dipole, the groundplane current I^d_ρ due to H^d_ϕ alone would be

(13) $$I^d_\rho\,(\rho) = -\,2\pi\rho H^d_\phi\,(z=0) = (I_0/2)\ ie^{ik\sqrt{\rho^2+(\lambda/4)^2}}$$

The current (13) flows on the groundplane regardless of whether it is infinite or finite. But when it is infinite, and only then, this current is just half of the total groundplane current, as follows from (5). However at the point $\rho = 0$, the total current $I_\rho(\rho)$ should be equal to the antenna current, regardless of groundplane size, and should have the value $-I_0$ according to (7). But

(14) $$I^d_\rho\,(\rho = 0) = -\,I_0/2\ .$$

Consequently at this point I^g_ρ and H^g_ρ are known and must have the values $(-I_0/2)$ and $(I_0/4\pi\rho)$, respectively. H^g_ϕ has a singularity at that point, whose form we now know exactly, even for finite groundplanes.

3. The Wave Equation and the Spheroidal Eigenfunctions.

Maxwell's equations in free space and for steady state fields,

(15) $$\nabla\times\underset{\sim}{E} = i\omega\mu_0\,\underset{\sim}{H}\ ,\quad \nabla\times\underset{\sim}{H} = -i\omega\,\varepsilon_0\underset{\sim}{E}\ ,$$

reduce to the vector wave equation

(16) $$\nabla\times\nabla\times\underset{\sim}{V} - k^2\,\underset{\sim}{V} = 0,$$

where $\underset{\sim}{V}$ is either field vector.

For reasons of symmetry both the simple resolution of these vectors, which is found in the previous section for dipoles alone, and the ϕ-independence of all components, are retained in the more complicated system consisting of the dipole and a circular groundplane. Thus

(17) $$\underset{\sim}{H} = \underset{\sim}{\phi}_1\,H_\phi\,(\rho,z)\ ,$$

and equation (16), when $\underset{\sim}{V} = \underset{\sim}{H}$, has only one scalar component equation, namely

$$(18) \qquad L\left[H_\phi\right] = \frac{\partial}{\partial\rho}\,\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\rho H_\phi + \frac{\partial^2}{\partial z^2}H_\phi + k^2 H_\phi = 0 .^*$$

It is now possible to formulate the mathematical problem of the present paper precisely. We look for the solution H_ϕ of (18) such that

$$(19) \qquad H_\phi = H_\phi^d + H_\phi^g ,$$

where H_ϕ^d is the known function (11c), subject to the following two boundary conditions. First, the condition at the surface of the perfectly conducting disk $\xi = 0$, where ξ is the radial oblate spheroidal coordinate[2]. This condition is that the normal derivative of the covariant components of H tangent to $\xi = 0$ must vanish[8]. In this problem it takes the simple form

$$(20) \qquad \frac{\partial H_\phi}{\partial\xi} = 0 , \qquad \xi = 0 .$$

Secondly, H_ϕ must satisfy Sommerfeld's radiation condition as $r \to \infty$, i.e., it must be a diverging wave of radial variation e^{ikr}/r .

The oblate spheroidal geometry enters through (20), and it is convenient to transform (18) to this system. Now

$$(21a) \qquad \rho = a\sqrt{(1-\eta^2)\,(1+\xi^2)} ,$$

$$(21b) \qquad \phi = \phi ,$$

$$(21c) \qquad z = a\,\eta\xi ,$$

where $0 \le \xi \le \infty$, $-1 \le \eta \le 1$, $0 \le \phi \le 2\pi$ and a is the radius of the disk $\xi = 0$. The wave equation (18) becomes

$$(22)\ L[H_\phi] =$$

$$= \frac{1}{a^2(\eta^2+\xi^2)}\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}(1+\xi^2)\frac{\partial}{\partial\xi} - \left(\frac{1}{1-\eta^2} - \frac{1}{1+\xi^2}\right) + k^2a^2(\eta^2+\xi^2)\right]H_\phi(\eta,\xi) = 0.$$

The variables η, ξ are separable. Let $\psi(\eta,\xi)$ be solutions of (22), and let

$$(23) \qquad \psi(\eta,\xi) = u_{\ell l}(\eta)\, v_{\ell l}(\xi).$$

---

* It should be noted that this is not the scalar wave equation $(\nabla^2 + k^2)H_\phi = 0$.

Then $u_{\ell 1}$ and $v_{\ell 1}$ satisfy the ordinary differential equations

$$\frac{d}{d\eta}(1-\eta^2)\frac{du_{\ell 1}}{d\eta} - \left[\frac{1}{1-\eta^2} - \alpha_{\ell 1} + k^2a^2(1-\eta^2)\right] u_{\ell 1} = 0\,, \tag{24a}$$

$$\frac{d}{d\xi}(1+\xi^2)\frac{dv_{\ell 1}}{d\xi} + \left[\frac{1}{1+\xi^2} - \alpha_{\ell 1} + k^2a^2(1+\xi^2)\right] v_{\ell 1} = 0, \tag{24b}$$

respectively, where $\alpha_{\ell 1}$ is the separation constant. The solutions ψ are a complete and orthogonal set of eigenfunctions, with the norm $N_{\ell 1} = \int_{-1}^{1} u_{\ell 1}^2(\eta)\,d\eta$, the *oblate spheroidal wave functions of the first order*. They are generated by the discrete eigenvalues $\alpha_{\ell 1}(ka)$, $\ell = 1,2,3,......$, which the separation constant is restricted to assume for every given ka. The condition which leads to eigenvalues is a physical one, namely that these functions must be finite in physical space, including in particular the values ± 1 for η. Inspection of (21a) shows that these values refer to the z axis.

Let us write (24a) in the form

$$u''(\eta) + e(\eta)\,u'(\eta) + f(\eta)\,u(\eta) = 0\;. \tag{25}$$

Whenever $e(\eta)$ and/or $f(\eta)$ have singularities for a given η (as happens here when $\eta = \pm 1$) such values of the variables are singular points of the equation, by definition. The condition (that only analytic solutions of the differential equation are admissible in physical space) produces these eigenvalues when it is applied at the singular points of equation (24a).

The index ℓ, introduced above, enumerates the eigenvalues and functions, while the index 1 refers to the fact that we deal with the functions of order one.

Once the $\alpha_{\ell 1}$ and $u_{\ell 1}(\eta)$ are found the radial solutions $v_{\ell 1}(\xi)$ are completely determined by (24b), since all parameters in this differential equation now are known.

We shall not discuss this set of special functions in detail here. The functions of *any* order, and in the notation used here, are discussed in a paper by Leitner and Spence, which will appear in the Journal of the Franklin Institute in April, 1950. There also exist recent papers by Stratton et al.[9], and Meixner[10]. A summary of the zero order functions, which are similar to the first order functions, is given in a recent report by the author.[11]

4. Green's Function

To the wave equation $L[\psi] = 0$ belongs a Green's function $G(P,Q)$, which is by definition symmetric in the arguments of the points P,Q, and a solution of

$$L_P\left[G(P,Q)\right] = -\frac{\delta(x_1-x_1')\delta(x_2-x_2')\delta(\phi-\phi')}{h_1(P)h_2(P)h_\phi(P)} \tag{26}$$

Here (x_1, x_2, ϕ) and (x_1', x_2', ϕ') are the coordinates of P and Q respectively in an orthogonal system, which is here taken to be one containing the azimuthal coordinate ϕ, and in which $h_1, h_2 h_\phi$ are the metric coefficients.

The solution ψ of the homogeneous equation is then given in terms of the boundary values of ψ and $\frac{\partial \psi}{\partial n}$ by

$$\psi(P) = -\int_\sigma \left[\psi(Q)\frac{\partial}{\partial n}G(P,Q) - \frac{\partial \psi(Q)}{\partial n} G(P,Q)\right] d\sigma_Q \tag{27}$$

These basic statements on solution by Green's function are widely discussed in the literature. Equations (26) and (27) agree with the formulation in a text by Morse and Feshbach[12]. If the region bounded by σ is infinite as in our problem the integration nevertheless is confined to the finite parts of the boundary provided both ψ and G obey the radiation condition as $r \to \infty$.

Consequently,

$$\sigma = \text{dipole} + \text{groundplane.}$$

Let ψ be the total magnetic field H_ϕ of our system. If in addition we require that

$$\frac{\partial G}{\partial n} = 0 \text{ on the groundplane,} \tag{28}$$

then the integral (27) is reduced to the dipole alone, because of the boundary condition (20) upon H_ϕ.

(29)
$$H_\phi(\eta,\xi) = -\int_d \left[H_\phi(\eta',\xi')\frac{\partial G(\eta,\xi;\eta',\xi')}{\partial n'} - \frac{\partial H_\phi(\eta',\xi')}{\partial n'} G(\eta,\xi;\eta',\xi')\right] d\sigma_d .$$

To complete the solution, therefore, we must determine G and evaluate this integral. Green's function is a solution of the homogeneous equation $L[\psi] = 0$ everywhere except when the points P and Q coincide. The problem of the homogeneous equation is the eigenvalue problem which we discussed in the previous

article. As outlined there it was solved in such a manner that the eigenvalues arose through a condition imposed upon the η - dimension. It leads to the mutually orthogonal eigenfunctions $u_{\ell 1}$ (η), $\ell = 1, 2, \ldots$, whose norms are $N_{\ell 1}$. Now Green's function is a symmetric function of its arguments, so a representation in the normalized spheroidal wave functions has the form

$$(30) \qquad G(\eta, \xi ; \eta' \xi') = \frac{1}{2\pi} \sum_{\ell=1}^{\infty} f_{\ell}(\xi, \xi') \; \frac{u_{\ell 1}(\eta) u_{\ell 1}(\eta')}{N_{\ell 1}} \quad .$$

While G is to be normalized in the space η, ξ, ϕ, it is associated to the ϕ -independent differential operator and L. Consequently it was taken to be a function of η and ξ only, but normalized in the ϕ -dimension by the factor $1/2\pi$.

We calculate f_{ℓ} through (26). In the spheroidal coordinates

$$(31) \qquad h_{\eta} \; h_{\xi} \; h_{\phi} = a^3(\eta^2 + \xi^2)^{*} \quad .$$

so the inhomogeneous term of (26) becomes

$$- \frac{\delta(\eta - \eta') \, \delta(\xi - \xi') \, \delta(\phi - \phi')}{a^3(\eta^2 + \xi^2)} \quad .$$

The differential expression L in spheroidal coordinates is given in (22). Thus

(32)

$$\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}(1 + \xi^2)\frac{\partial}{\partial \xi} - (\frac{1}{1-\eta^2} - \frac{1}{1+\xi^2}) + k^2 a^2(\eta^2 + \xi^2) \quad G(\eta, \xi ; \eta', \xi') =$$

$$= - \delta(\eta - \eta') \, \delta(\xi - \xi') \delta(\phi - \phi') / \; a \quad .$$

We substitute (30), remembering the differential equation (24a) for $u_{\ell 1}$.

(33)

$$\frac{1}{2\pi} \sum_{\ell=1}^{\infty} \frac{u_{\ell 1}(\eta) u_{\ell 1}(\eta')}{N_{\ell 1}} \left[- \alpha_{\ell 1} + \frac{\partial}{\partial \xi}(1 + \xi^2)\frac{\partial}{\partial \xi} + \frac{1}{1+\xi^2} + k^2 a^2 (1+\xi^2) \right] f_{\ell}(\xi, \xi') =$$

$$= - \delta(\eta - \eta') \delta(\xi - \xi') \delta(\phi - \phi') / a$$

Multiply by one of the set of orthogonal functions $(u_{\ell 1})$, and integrate with respect to η and ϕ over the ranges of these variables in space

(34)

$$\left[\frac{\partial}{\partial \xi}(1 + \xi^2)\frac{\partial}{\partial \xi} + \frac{1}{1+\xi^2} - \alpha_{\ell 1} + k^2 a^2 (1+\xi^2) \right] f_{\ell}(\xi ; \xi') = - \delta(\xi - \xi')/a \; .$$

This is the equation for Green's function in the ξ dimension. Solutions of its homogeneous counterpart (24b) were in the previous section described as the radial spheroidal wave functions $v_{\ell 1}(\xi)$. Thus f_ℓ is built up of these. Now it is known from the general theory that f_ℓ is both symmetric and continuous[13] (i.e. across $\xi = \xi'$), and therefore it is a product of solutions $v_{\ell 1}$.

The boundary conditions which serve to determine f_ℓ have been cited previously. As ξ or $\xi' \to \infty$ it must behave as a diverging wave (Sommerfeld's radiation condition). This is met by the radial wave functions of the third kind, described in reference 11,

$$ {}^{(3)}v_{\ell 1} = {}^{(1)}v_{\ell 1} + i\,{}^{(2)}v_{\ell 1} \tag{35}$$

where ${}^{(1)}v_{\ell 1}$ and ${}^{(2)}v_{\ell 1}$ are a known fundamental set of independent solutions of (24b), also described there.

At the groundplane (28) must hold. We define another linear combination of the fundamental set, say, ${}_1v_{\ell 1}$

$$ {}_1v_{\ell 1} = a_\ell\, {}^{(1)}v_{\ell 1} + b_\ell\, {}^{(2)}v_{\ell 1} \tag{36}$$

where a_ℓ and b_ℓ are constants such that (28) is satisfied.

$$ 0 = a_\ell\, {}^{(1)}v_{\ell 1}'(0) + b_\ell\, {}^{(2)}v_{\ell 1}'(0). \tag{37}$$

This determines the constants only to within a factor, e.g., we must look for a second linear equation involving them.

Before doing so, note that $f_\ell(\xi, \xi')$ is now known (in terms of a_ℓ and b_ℓ, that is, which are still unknown)

$$ f_\ell = \begin{cases} {}_1v_{\ell 1}(\xi')\ {}^{(3)}v_{\ell 1}(\xi), & \xi \geq \xi', \\ {}_1v_{\ell 1}(\xi)\ {}^{(3)}v_{\ell 1}(\xi'), & \xi' \geq \xi, \end{cases} \tag{38}$$

for the expression in braces is both symmetric in ξ, ξ' and continuous at $\xi = \xi'$.

To complete the calculation, we need only to use the condition of discontinuity at ξ' upon the derivative $\partial f_\ell / \partial \xi$, which follows from (34). Let a discontinuity at ξ' be denoted by $[\ \ ]_{\xi'}$. If (34) is integrated with respect to ξ over an interval ϵ including the point ξ', and one finally lets

$\epsilon \to 0$, one obtains

$$\left[\frac{\partial f_\ell}{\partial \xi}\right]_{\xi'} = -\frac{1}{a(1+\xi'^2)} , \tag{39}$$

because the terms on the left hand side of (34) not involving derivatives of f_ℓ are continuous.

As is shown in a separate paper, mentioned previously[14], the Wronskian of ${}^{(1)}v_{\ell 1}(\xi')$, ${}^{(2)}v_{\ell 1}(\xi')$ is

$$W_{\ell 1}(\xi') = \frac{W_{\ell 1}(0)}{1+\xi'^2} , \tag{40}$$

where $W_{\ell 1}(0)$ is a known constant (with respect to ξ').* The lefthand side of (39) becomes, by combining (38) with (35) and (36),

$${}_1v_{\ell 1}(\xi')\,{}^{(3)}v'_{\ell 1}(\xi') - {}_1v'_{\ell 1}(\xi')\,{}^{(3)}v_{\ell 1}(\xi') = (ia_\ell - b_\ell)\,W_{\ell 1}(\xi'). \tag{41}$$

A prime on $v_{\ell 1}$ denotes the derivative. Thus (39) can be rewritten:

$$ia_\ell - b_\ell = -\frac{1}{a\,W_{\ell 1}(0)} . \tag{42}$$

The simultaneous solution of (37) and (42) is

$$a_\ell = \frac{-{}^{(2)}v'_{\ell 1}(0)}{aW_{\ell 1}(0)\,{}^{(3)}v'_{\ell 1}(0)} , \quad b_\ell = \frac{{}^{(1)}v'_{\ell 1}(0)}{aW_{\ell 1}(0)\,{}^{(3)}v'_{\ell 1}(0)} . \tag{43}$$

Now f_ℓ is completely determined. Making use of the footnote below, and defining the function

$$\begin{aligned} {}_2v_{\ell 1}(\xi') &= aW_{\ell 1}(0)\,{}^{(3)}v_{\ell 1}(0)\cdot {}_1v_{\ell 1}(\xi'), \\ &= {}^{(2)}v'_{\ell 1}(0)\,{}^{(1)}v_{\ell 1}(\xi') - {}^{(1)}v'_{\ell 1}(0)\,{}^{(2)}v_{\ell 1}(\xi'), \end{aligned} \tag{44}$$

we have finally

$$\begin{aligned} G(\eta,\xi;\eta',\xi') &= \\ &= \frac{k}{2\pi}\sum_{\ell=1}^{\infty} \frac{(-)^\ell}{N_{\ell 1}(2q_{\ell 1}/ka)^2\,{}^{(3)}v'_{\ell 1}(0)}\,u_{\ell 1}(\eta)\,u_{\ell 1}(\eta') \begin{cases} {}_2v_{\ell 1}(\xi')\,{}^{(3)}v_{\ell 1}(\xi), & \xi \geq \xi', \\ {}_2v_{\ell 1}(\xi)\,{}^{(3)}v_{\ell 1}(\xi'), & \xi' \geq \xi . \end{cases} \end{aligned} \tag{45}$$

---

*$W_{\ell 1}(0) = (-)^{\ell-1}(2q_{\ell 1}/ka)^2/ka$, where $q_{\ell 1}$ is a known constant which arises from the normalization chosen for the radial functions; it depends on ℓ and on ka.

5. Derivation of the Total Field

We next calculate the total magnetic field H_ϕ by evaluating the integral (29), which is taken along the dipole. This is a portion of the z'axis, $0 \leq z' \leq \frac{\lambda}{4}$, and $\eta' = 1$ there. The integration variable therefore is $\xi' = z'/a$, as follows from (21c); since $k\lambda = 2\pi$,

$$0 \leq \xi' \leq \frac{\pi}{2ka} .$$

The integration is complicated by apparent singularities in the integrand as $\eta' \to 1$. For example,

$$(46) \qquad H^d_\phi\,(\eta',\xi') \to \frac{I_0}{2\pi a}\,\frac{\cos(ka\,\xi')}{(1-\eta'^2)^{1/2}\,(1+\xi'^2)^{1/2}} \quad \text{when } \eta' \to 1;$$

this is found when the expression (11c) for H^d_ϕ is transformed to spheroidal coordinates by use of equations (21), and when one lets $\eta' \to 1$.*

We shall show, however, that the integrand is finite term by term in the representation by eigenfunctions as $\ell = 1,2,\ldots.$ It is best to do so by evaluating the integrand which is defined over a segment of the z'axis, as a limit, as $\eta' \to 1$, of another integrand: one which is defined on a surface $\eta \neq 1$ but close to 1. This is a section of a hyperboloid[2] between $\xi' = 0$ and $\xi' = \pi/2ka$. The surface element on the hyperboloid

$$(47 \qquad d\,\sigma_d = h_{\xi'}\,h_{\phi'}\,d\xi'\,d\phi' .$$

In the operator $\partial/\partial n'$, the direction of the normal is to be taken as positive into the hyperboloid**; i.e., along increasing η',

$$(48) \qquad \frac{\partial}{\partial n'} = \frac{\partial}{h_{\eta'}\,\partial\eta'} .$$

---

*If the same procedure were applied to the calculating $\partial H^d_\phi/\partial\eta'$, one would obtain

$$(46a) \qquad \frac{\partial H^d_\phi}{\partial\eta'} \to \frac{I_0}{2\pi a}\,\frac{\cos(ka\xi')}{(1-\eta'^2)^{3/2}\,(1+\xi'^2)^{1/2}} + - - -, \text{ as } \eta' \to 1$$

where the dots (- - -) denote terms whose singularity in $(1-\eta'^2)$ is of a degree less than 3/2; they can be neglected for reasons which will be clear later.

**Because it is understood to point out of the space in which the field is sought.

Now

(49) $$\frac{d\sigma_d}{h_{\eta'}} = (h_{\xi'} h_{\phi'} / h_{\eta'})\, d\xi' d\phi' = a(1-\eta'^2) d\xi' d\phi' ,$$

for, from a knowledge of our coordinate system[11], one finds that

(50) $$h_{\eta'} = a\left(\frac{\eta'^2+\xi'^2}{1-\eta'^2}\right)^{1/2} \quad h_{\xi'} = a\left(\frac{\eta'^2+\xi'^2}{1+\xi'^2}\right)^{1/2} , \quad h_{\phi'} = a(1-\eta'^2)^{1/2}(1+\xi'^2)^{1/2} \quad .$$

The integral now has the form

(51)
$$H_\phi(\eta,\xi) = -\int_0^{2\pi} \int_0^{\frac{\pi}{2ka}} \left[(H_\phi \frac{\partial G}{\partial \eta'} - \frac{\partial H_\phi}{\partial \eta'} G)(1-\eta'^2)\right]_{\eta' \to 1} a d\xi' d\phi'$$

We next must show that no part of the bracketed expression has a singularity in $(1-\eta'^2)$. The η-behavior of G is from (45) clearly the behavior of $u_{\ell 1}(\eta)$, term by term. But

(52) $$u_{\ell 1} = (1-\eta'^2)^{1/2}\left[1 + c_2^{\ell 1}(1-\eta'^2) + c_4^{\ell 1}(1-\eta'^2)^2 + \dots\right],$$ [12]

where $c_2^{\ell 1}$, $c_4^{\ell 1}$, ... are knowncoefficients. And

(53) $$\frac{du_{\ell 1}}{d\eta'} = -\frac{\eta'}{(1-\eta'^2)^{1/2}}\left[1 + 3c_2^{\ell 1}(1-\eta'^2) + 5c_4^{\ell 1}(1-\eta'^2)^2 + \dots\right].$$

(Equivalent results arise when the functions $u_{\ell 1}$ are odd in η')

Thus (52) represents the dependence of G upon η' and (53) does the same for $\partial G/\partial\eta'$.

As regards H_ϕ, we remember that

(19) $$H_\phi = H_\phi^d + H_\phi^g \quad .$$

Let us first consider H_ϕ^g, which is the field due to the groundplane, i.e. the field set up in the groundplane by the dipole. Because of the linearity of the wave equation, H_ϕ^g must itself be a solution of

(54) $$L[\psi] = 0 \; .$$

It follows from section 3 that H_ϕ^g is a series of terms like $u_{\ell 1}(\eta)\, v_{\ell 1}(\xi)$. Therefore its behavior in η', and that of its η'-derivative is also given by

(52) and (53). Consequently

$$(55)\quad \left.\begin{array}{r} H^{g}_{\phi}\dfrac{\partial G}{\partial \eta'}\cdot(1-\eta'^{2}) \\[2ex] \dfrac{\partial H^{g}}{\partial \eta'}\, G\cdot(1-\eta'^{2}) \end{array}\right\} \sim (1-\eta'^{2}) + \mathcal{O}(1-\eta'^{2})^{2} \longrightarrow 0 \;, \quad \text{as}\ \eta' \to 1,$$

and we are left with

$$(56)\quad H_{\phi}(\eta\,\xi\,) = -\int_{0}^{2\pi}\int_{0}^{\frac{\pi}{2ka}}\left[\left(H^{d}_{\phi}\frac{\partial G}{\partial \eta'} - \frac{\partial H^{d}_{\phi}}{\partial \eta'}\,G\,\right)(1-\eta'^{2})\right]_{\eta'\to 1} a\,d\xi'\,d\phi' .$$

Because of (46) and (46a), the latter given in a footnote to page 15, the term by term dependence upon η' of the expressions in the integrand of (56) are of the form

$$(57)\quad \left.\begin{array}{l} H^{d}_{\phi}\,\dfrac{\partial u_{\ell 1}}{\partial \eta'}\cdot(1-\eta'^{2}) = \dfrac{-I_{0}}{2\pi a}\,\dfrac{\cos(ka\xi')}{(1+\xi'^{2})^{1/2}}\,, \\[3ex] \dfrac{\partial H^{d}_{\phi}}{\partial \eta'}\,u_{\ell 1}\cdot(1-\eta'^{2}) = \dfrac{I_{0}}{2\pi a}\,\dfrac{\cos(ka\xi')}{(1+\xi'^{2})^{1/2}}\,, \end{array}\right\}\ \text{when}\ \eta' = 1;$$

all other terms contain $(1-\eta'^{2})$ to a positive exponent and vanish for $\eta' = 1$.

As a consequence we obtain the following solution, after substituting (45) into (56): when $\xi \geqslant \xi'$,

$$(58)\quad H_{\phi} = \frac{kI_{0}}{\pi}\sum_{\ell=1}^{\infty}\left\{\frac{(-)^{\ell}\displaystyle\int_{0}^{\pi/2ka}\left[{}_{2}v_{\ell 1}(\xi')\cos(ka\xi')\,d\xi'/(1+\xi')^{1/2}\right]}{N_{\ell 1}(2q_{\ell 1}/ka)^{2}\ {}^{(3)}v'_{\ell 1}(0)}\right\}u_{\ell 1}(\eta)\ {}^{(3)}v_{\ell 1}(\xi),$$

and when $\xi = 0$, i.e. on the groundplane, the only value of ξ such that $\xi' \geqslant \xi$, because $0 \leq \xi' \leq \pi/2ka$,

$$(59)\quad H_{\phi}(\eta,0) = \frac{kI_{0}}{\pi}\sum_{\ell=1}^{\infty}\left\{\frac{(-)^{\ell}\ {}_{2}v_{\ell 1}(0)\displaystyle\int_{0}^{\pi/2ka}\left[{}^{(3)}v_{\ell 1}(\xi')\cos(ka\xi')\,d\xi'/(1+\xi'^{2})^{1/2}\right]}{N_{\ell 1}(2q_{\ell 1}/ka)^{2}\ {}^{(3)}v'_{\ell 1}(0)}\right\}u_{\ell 1}(\eta) .$$

We thus have obtained the solution of our problem. However, it can be put into simpler form. It can be shown, first of all, that

$$(60)\quad (-)^{\ell-1}\ {}_{2}v_{\ell 1}(0)/(2q_{\ell 1}/ka)^{2} = 1/ka\ {}^{*}$$

---

*Proof of this relationship is omitted; it is based on a knowledge of the radial functions, which are discussed in reference 14.

Furthermore, it is of physical interest here to compute the total field in two different regions: on the groundplane, in order to calculate the surface currents; and at large distances, i.e., when $\xi \to \infty$, in order to find the radiation properties. As a consequence, (58) can be simplified, for

(61) $$ {}^{(3)}v_{\ell 1}(\xi) = -(2q_{\ell 1}/k^2a)\, e^{ikr}/r \quad \text{as } \xi \to \infty.^{*} $$

Finally, the integrals of type

(62) $$ I_\ell = \int_0^{\pi/2ka} \left[v_{\ell 1}(\xi') \cos(ka\,\xi')\, d\xi' / (1+\xi'^2)^{1/2} \right] $$

can be evaluated explicitly. We shall denote them by ${}_2I_\ell$ or ${}^{(3)}I_\ell$ depending on whether ${}_2v_{\ell 1}$ or ${}^{(3)}v_{\ell 1}$ are involved, respectively. We introduce the functions

(63) $$ w_{\ell 1}(\xi') = (1+\xi'^2)^{1/2}\, v_{\ell 1}(\xi') . $$

The differential equation (24b) transformed to the $w_{\ell 1}$ is simply

(64) $$ w''_{\ell 1} + k^2a^2 w_{\ell 1} = \alpha_{\ell 1} w_{\ell 1} / (1+\xi'^2). $$

In conjunction with (64) consider the harmonic equation

(65 $$ y'' + k^2a^2 y = 0. $$

It follows that

(66) $$ y w''_{\ell 1} - y'' w_{\ell 1} = \alpha_{\ell 1} y w_{\ell 1} / (1+\xi'^2), $$

(67) $$ \Big| y w'_{\ell 1} - y' w_{\ell 1} \Big|_{\xi_1}^{\xi_2} = \alpha_{\ell 1} \int_{\xi_1}^{\xi_2} \left[y w_{\ell 1}\, d\xi' / (1+\xi'^2) \right]. $$

The function $\cos(ka\,\xi')$ is a solution of (65). Consequently

(68) $$ \int_{\xi_1}^{\xi_2} \frac{v_{\ell 1} \cos(ka\,\xi')\, d\xi'}{(1+\xi'^2)^{1/2}} = \frac{\Big| w'_{\ell 1} \cos(ka\,\xi') + ka\, w_{\ell 1} \sin(ka\,\xi') \Big|_{\xi_1}^{\xi_2}}{\alpha_{\ell 1}} , $$

and we get the results

(69) $$ {}_2I_\ell = ka\, {}_2w_{\ell 1}(\pi/2ka) / \alpha_{\ell 1} , $$

(70) $$ {}^{(3)}I_\ell / {}^{(3)}v'_{\ell 1}(0) = -\left[1 - ka\, {}^{(3)}w_{\ell 1}(\pi/ka) / {}^{(3)}w'_{\ell 1}(0) \right] / \alpha_{\ell 1} , $$

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* See footnote page 17.

where we made use of the identity ${}^{(3)}v'_{\ell 1}(0) = {}^{(3)}w'_{\ell 1}(0)$ in (70).

We may now write the results in final form:

$$\boxed{\begin{aligned} &(71) \qquad H_\phi = \frac{I_0 k^2 a^2}{2\pi} \frac{e^{ikr}}{r} \left[\sum_{\ell=1}^{\infty} A_\ell \, u_{\ell 1}(\eta) \right], \quad r \to \infty , \\ &(72) \qquad H_\phi = \frac{I_0}{\pi a} \left[\sum_{\ell=1}^{\infty} B_\ell \, u_{\ell 1}(\eta) \right] , \quad \xi = 0 , \end{aligned}}$$

where

$$(73) \qquad A_\ell = \frac{(-)^{\ell-1} {}_2 w_{\ell 1}(\pi/2ka)}{\alpha_{\ell 1} N_{\ell 1} q_{\ell 1} {}^{(3)}w'_{\ell 1}(0)} ;$$

$$(74) \qquad B_\ell = \frac{1 - ka \, {}^{(3)}w_{\ell 1}(\pi/2ka) / {}^{(3)}w'_{\ell 1}(0)}{\alpha_{\ell 1} N_{\ell 1}} .$$

6. The Current Distribution on the Groundplane

If $\mathbf{n}_\xi$ is the unit vector normal to the spheroid of constant ξ, then the surface current density is

$$(3) \qquad \mathbf{K} = \mathbf{n}_\xi \times \mathbf{H} , \quad \xi = 0 ,$$

as already noted. The coordinate system is righthanded in the order η, ξ, ϕ, so if $\mathbf{n}_\phi$, $\mathbf{n}_\eta$ are the remaining unit vectors,

$$(75) \qquad \mathbf{K} = \mathbf{n}_\xi \times \mathbf{n}_\phi \cdot H_\phi = \mathbf{n}_\eta H_\phi .$$

$\mathbf{K}$ points in the direction of increasing η, which is radial on the circle $\xi = 0$; $\eta = -1$ at the center point of the bottom side of the groundplane, $\eta \to 0$ radially outward to the edge and increases radially inward on the antenna side to $\eta = 1$ which is the center just under the antenna.

$$(76) \qquad K_\eta(\eta) = \frac{I_0}{\pi a} \sum_{\ell=1}^{\infty} B_\ell \, u_{\ell 1}(\eta) .$$

This result can be obtained directly by a method involving the reciprocity theorem: given the surface currents on one closed surface, such as on the dipole, it is possible to use this theorem in integral form to find

the current on a conductor in its field, say, the groundplane. The method was recently applied by Papas and King[15] to a dipole radiating above a sphere (whose radius is comparable to the wavelength), and the currents on the sphere are thus calculated. This alternate way to study the groundplane is given in the appendix to our report.

While (76) is the desired result, it is proper only if the series converges. This is not the case, and an analysis of (74) based on the eigenfunctions shows that the coefficients diverge slowly in the real part.

$$\mathrm{Re}(B_\ell) \approx \ell \,;$$

on the other hand $\mathrm{Im}(B_\ell)$ is rapidly convergent.

In other words, the divergence occurs in the part of K_η in phase with the antenna current, which is entirely real.* The divergence is due to a singularity in the real part of K_η which does not explicitly appear in its eigenfunction representation (76). Since the $u_{\ell 1}(\eta)$ are entire functions of η[14], (76) displays no singularity, term by term. However, if through other information a set of singularities is known, if these are isolated from the expansions by well-known methods, and if the remaining series can be shown to converge: then this known set is the complete set of singularities, and the result is a valid solution of the problem.

Thus we are brought back to the observations in section 2b. For present purposes we split K_η into

$$K_\eta^g(\eta) + K_\eta^d(\eta)$$

the parts due to H_ϕ^g and H_ϕ^d respectively on the groundplane. But

$$(12) \qquad H_\phi^d = -\frac{I_0}{4\pi}\,\frac{i e^{ik\sqrt{\rho^2 + (\lambda/4)^2}}}{\rho} \,,$$

$$(77) \qquad = -\frac{I_0}{4\pi a}\,\frac{i e^{ika\sqrt{1 + (\pi/2ka)^2 - \eta^2}}}{(1-\eta^2)^{1/2}} \,,$$

making use of (21a). I.e., it is singular at $|\eta| = 1$, where we must distinguish two points, the centers of the bottom and top sides of the groundplane. But H_ϕ^d exists independently of the conducting groundplane and is therefore the

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* The functions $u_{\ell 1}(\eta)$ are real.

same on both sides. This is clear also because (77) is an even function of η. Let $|\eta| \to 1$ in (77); then K^d_η is entirely real, e.g.

$$K^d_\eta(\pm 1) \to \frac{I_0}{4\pi a(1-\eta^2)^{1/2}} \cdot \tag{78}$$

As regards the current $K^g_\eta(\eta)$, one can show that it must be odd in η, that is, across the groundplane, by arguments based on the fact that the scatterer is plane. The proof was given in a previous report[16], for a different function, but satisfying the same boundary condition as our H_ϕ.

We now combine the result that K^g_η must be odd in η with the result already obtained at the end of section 2b; namely, that at the top center point $\eta = 1$ on the groundplane $H^d_\phi = H^g_\phi$. Thus we take the following singularity, odd in η,

$$K^g_\eta(\eta) \to \frac{I_0\,\eta}{4\pi a(1-\eta^2)^{1/2}} , \qquad |\eta| \to 1. \tag{79}$$

And as a result, the total current on the groundplane at $\eta = 1$

$$I_\eta = 2\pi\rho K_\eta = 2\pi a(1-\eta^2)^{1/2}(K^d_\eta + K^g_\eta) = I_0 , \tag{80}$$

but as $\eta \to -1$, that is, at the center point of the bottom side,

$$I_\eta \to 0 , \tag{81}$$

which is physically necessary for at this point there must be no source of current, as distinguished from the point at the center of the antenna side.

In order to isolate singularities (78),(79) from the expansion (76) for K_η, $1/(1-\eta^2)^{1/2}$ and $\eta/(1-\eta^2)^{1/2}$ will be expanded in terms of the even and odd subsets of the functions $u_{\ell 1}(\eta)$, respectively. These functions are even or odd in η when $(\ell - 1)$ is an even or odd [redacted] integer, respectively.

$$1/(1-\eta^2)^{1/2} = {\sum_{\ell=1}^{\infty}}' C_\ell \, u_{\ell 1}(\eta), \tag{81a}$$

$$\eta/(1-\eta^2)^{1/2} = {\sum_{\ell=2}^{\infty}}' C_\ell \, u_{\ell 1}(\eta) , \tag{81b}$$

a primed summation sign denoting summation over alternate integers ℓ. And the set of coefficients C_ℓ, which is necessarily divergent, is given by

$$C_\ell = \frac{1}{N_{\ell 1}} \int_{-1}^{1} u_{\ell 1}(\tau)\,d\tau/(1-\tau^2)^{1/2} , \quad \ell - 1 \ \text{even} , \tag{82a}$$

$$C_\ell = \frac{1}{N_{\ell 1}} \int_{-1}^{1} u_{\ell 1}(\tau)\,d\tau/(1-\tau^2)^{1/2} , \quad \ell - 1 \ \text{odd}. \tag{82b}$$

Accordingly

$$(83)\quad K_\eta(\eta) = \frac{I_0}{4\pi a}\left[\frac{1}{(1-\eta^2)^{1/2}} + \frac{\eta}{(1-\eta^2)^{1/2}} + \sum_{\ell=1}^{\infty}(4B_\ell - C_\ell)\, u_{\ell 1}(\eta)\right].$$

Computation shows that the set of coefficients $(4B_\ell - C_\ell)$ converges. Consequently no other singularities than the ones we have postulated exist in the current (or H_ϕ) on the groundplane: around the edge, where it might have been supposed that a singularity or at least discontinuity exists, the current is in fact continuous as one passes from the underside to the antenna side.

Finally, we have noted that the even part of the expression for K_η is completely known from the dipole theory. Thus the calculation is simplified if the dipole values are substituted. Let us also multiply by $2\pi\rho = 2\pi a(1-\eta^2)^{1/2}$ and thus calculate the (non-singular) total radial current, e.g.,

$$(84)\quad I_\eta(\eta) = (I_0/2)\left[-ie^{ik\sqrt{\rho^2+(\lambda/4)^2}} + \eta + {\sum_{\ell=1}^{\infty}}'(4B_\ell - C_\ell)\, u_{\ell 1}(\eta)\right].^{*}$$

The quantity $I_\eta(\eta)/I_0$ is plotted against $a(1-\eta^2)^{1/2}=\rho$ in figures 2 to 5 for four different values of ka, 3, 4, 5 and $\sqrt{42}$. The variable in this parameter is in effect a, the radius of the groundplane. Since I_η is continuous about the edge, its value on the bottom side is plotted by merely continuing the abscissa values to twice the radius of the groundplane.

It is clear from previous discussion that $\mathrm{Re}(I/I_0)$ is always unity at top center, whereas $\mathrm{Im}(I/I_0)$ must vanish there since the antenna current is entirely real. At bottom center the current always vanishes.

When $a \to \infty$, we already noted in section 2a that the absolute value of the topside current is I_0, while the bottomside current vanishes. From a study of the curves for $|I/I_0|$ in the figures we can draw the following conclusions: (i) as ka increases the average current on the bottomside decreases; (ii) the current on the topside oscillates about the value I_0. The results are thus consistent with the limiting case $a \to \infty$. At the edge $\eta=0$, however, the absolute value of I is always $I_0/2$. This also follows directly from (84), for the expression in the bracket becomes simply unity, since $u_{\ell 1}(0) = 0$ for odd $(\ell - 1)$.

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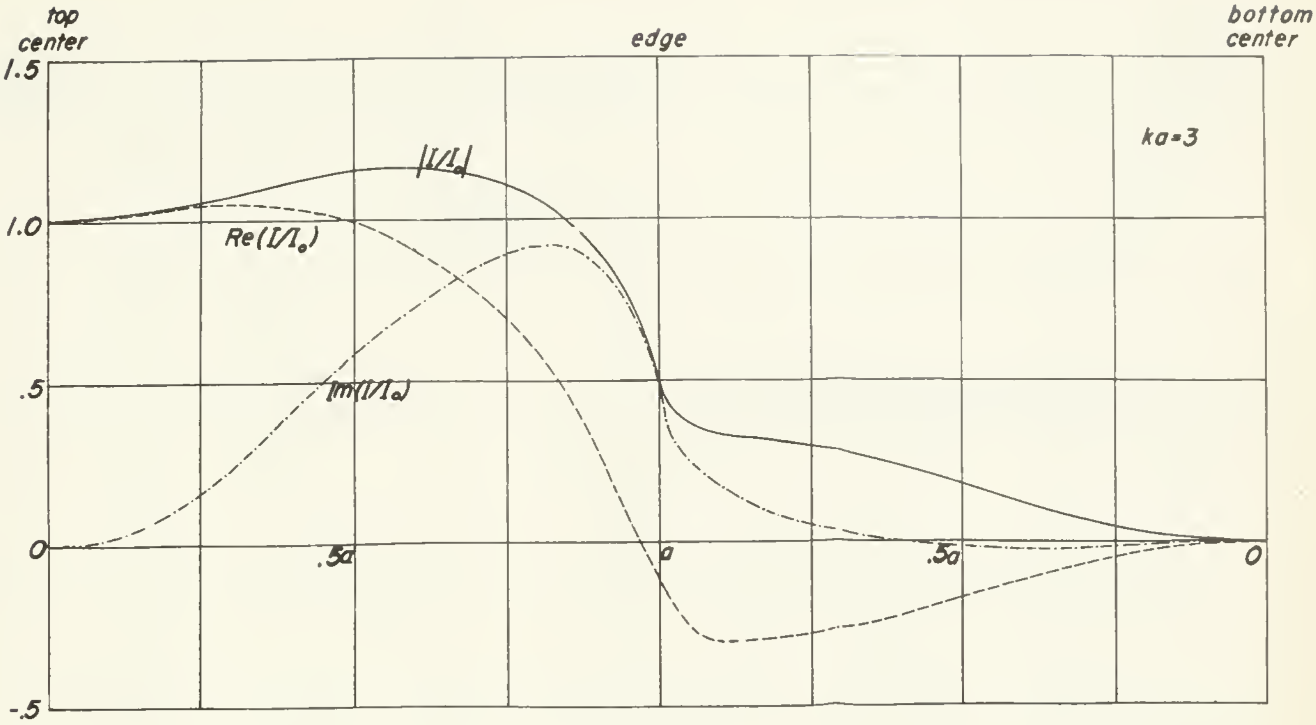
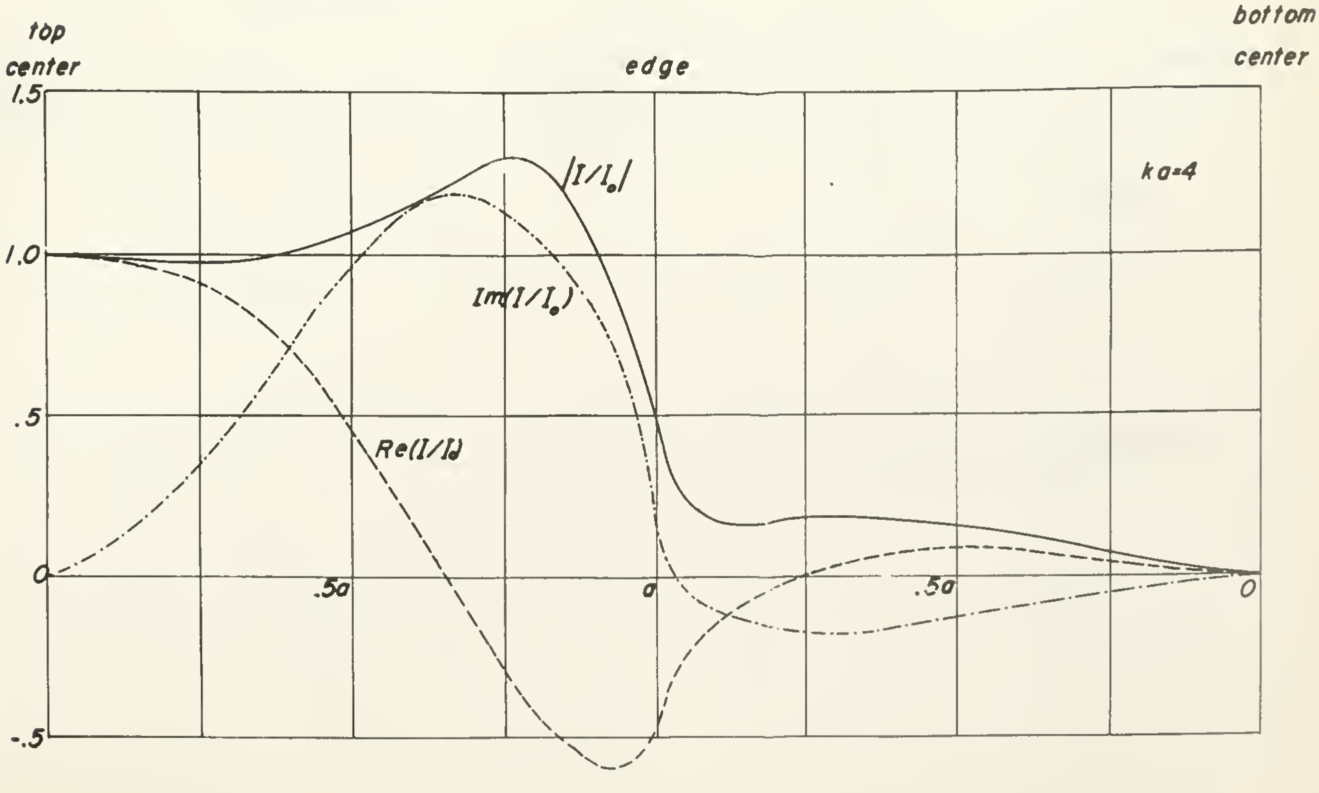
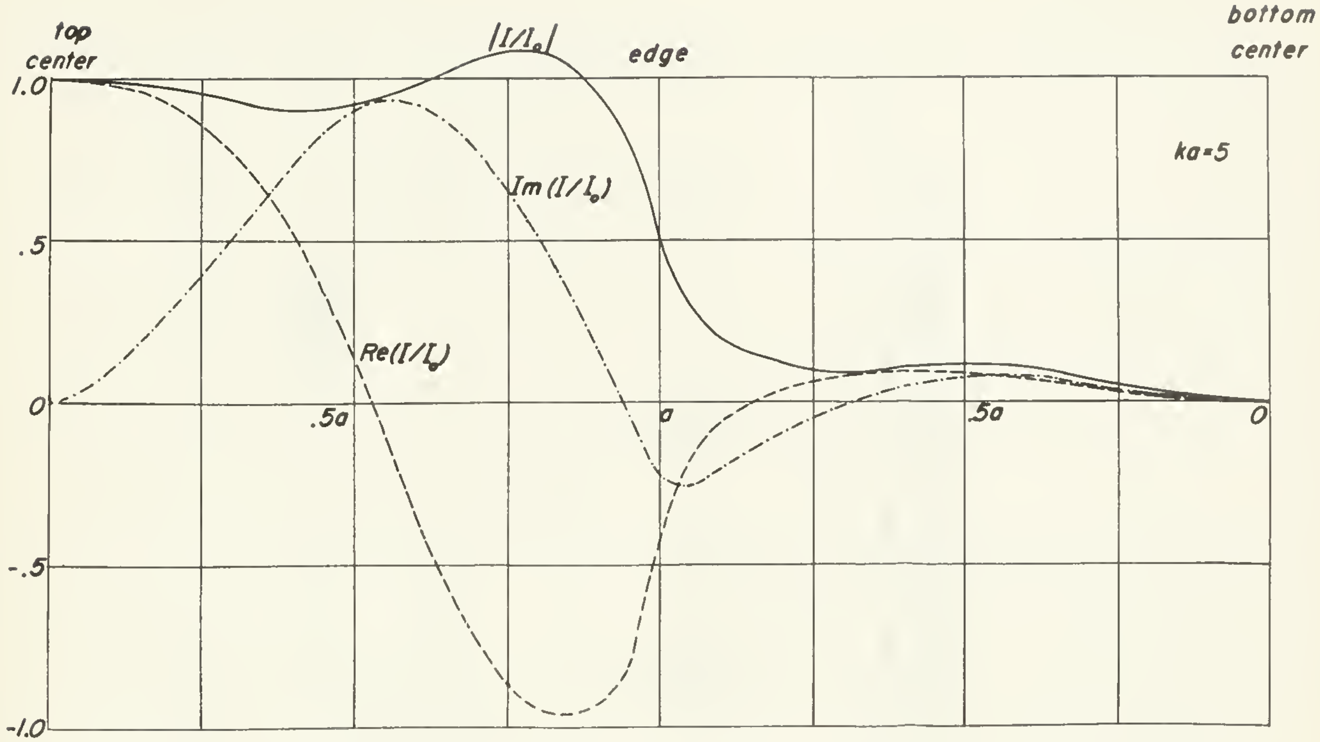
*Note that computation with spheroidal functions is limited to $(\ell - 1)$ odd. This is a considerable reduction in labor.

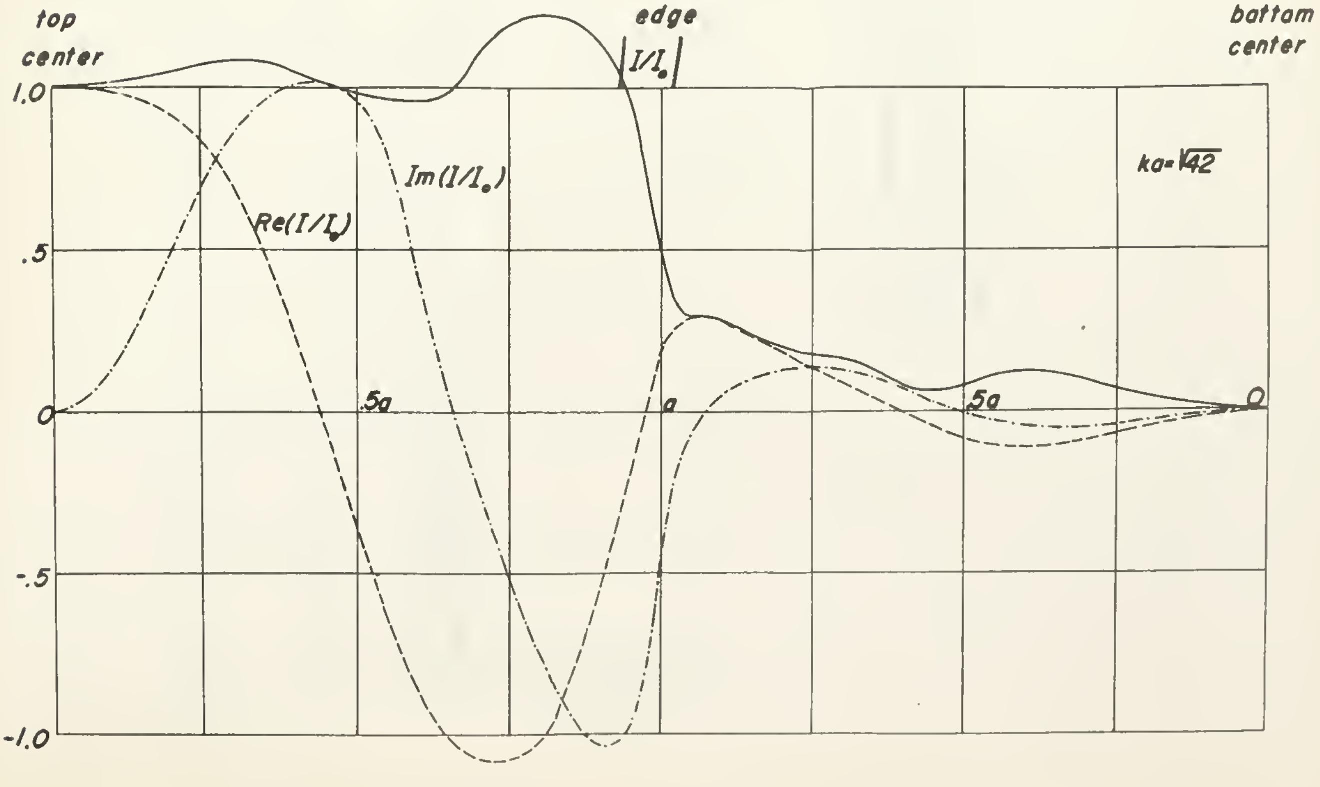


Figs. 2 and 3 - Ratio of total radial current I on groundplane to amplitude I_0 of dipole current, plotted against distance, when ka = 3 and ka = 4.





Figs. 4 and 5 - Ratio of total radial current I on groundplane to amplitude I_0 of dipole current, plotted against distance, when ka = 5 and ka = $\sqrt{42}$.



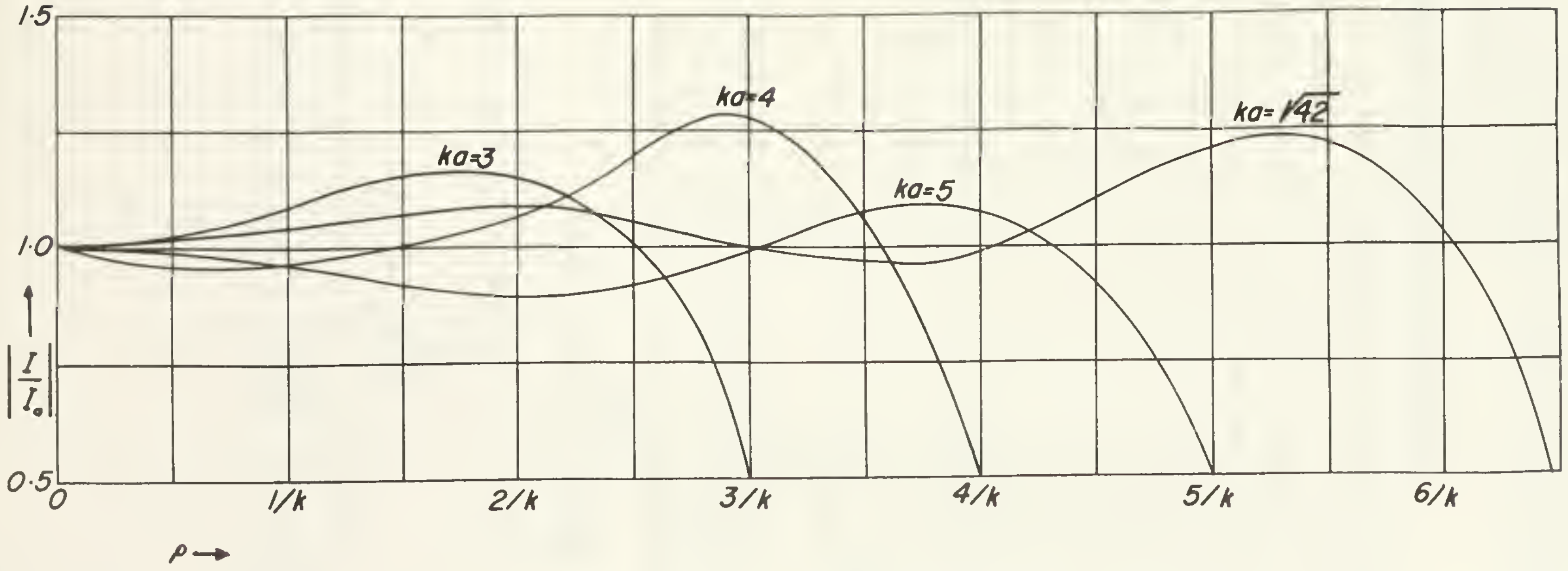


Fig. 6 - I/I_o on antenna side only, for four different values of ka, plotted to scale against distance from center. As ka increases, the average becomes more nearly unity.

Point (ii) just mentioned is illustrated by figure 6, in which $|I/I_0|$, on the antenna side, is plotted to scale against distance from the center, for the different values of ka corresponding to different values of the groundplane radius.

7. Radiation Properties of the System

We next investigate the field at large distances from the radiating system, starting with expression (71) for H_ϕ at such points. In particular we calculate the flux of power as a function of direction and its integral over all directions. In this way we will find the radiation pattern and resistance corresponding to various values of groundplane radius and will compare the results with the half-wave dipole field.

The radiation field of a finite current distribution is transverse to the radial direction. While obvious here for the magnetic vector, this fact must still be proved for $\underset{\sim}{E}$. It is easily demonstrated when $\underset{\sim}{E}$ is expanded in polar coordinates from the form given by the second of Maxwell's equations (15). Let $\underset{\sim}{n}_r$, $\underset{\sim}{n}_\theta$, $\underset{\sim}{n}_\phi$ be the polar unit vectors, then, since $\underset{\sim}{H} = \underset{\sim}{n}_\phi H_\phi$,

$$\underset{\sim}{E} = \frac{i}{\omega\epsilon_0} \nabla\times\underset{\sim}{H} = \frac{i}{\omega\epsilon_0}\left[\underset{\sim}{n}_r \frac{\partial(\sin\theta H_\phi)}{r\sin\theta\,\partial\theta} - \underset{\sim}{n}_\theta \frac{\partial(rH_\phi)}{r\,\partial r}\right] . \tag{85}$$

Remembering from (71) that the dependence of H_ϕ on r is as 1/r when $r\to\infty$, only one term in (85) will have this dependence on r, while the others decrease more rapidly with r and are negligible. This term is

$$-\frac{i}{\omega\epsilon_0}\underset{\sim}{n}_\theta \frac{\partial H_\phi}{\partial r} = \frac{k}{\omega\epsilon_0}\underset{\sim}{n}_\theta H_\phi . \tag{86}$$

The right hand side follows from (71). Now $k/\omega = 1/c = \sqrt{\epsilon_0\mu_0}$, so

$$E = E_\theta = \sqrt{\frac{\mu_0}{\epsilon_0}}\, H_\phi . \tag{87}$$

Consequently the Poynting flux is, ($\tilde{H}_\phi$ denotes complex conjugate)

$$S^* = \frac{1}{2} E_\theta \tilde{H}_\phi = \frac{I_0^2}{8\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{r^2}(ka)^4\left|\sum^{\infty} A_\ell u_{\ell 1}(\eta)\right|^2 . \tag{88}$$

This function of angular orientation η *is the radiation pattern of energy flux for the groundplane system, which is to be compared to the pattern above an infinite groundplane given by (8) and drawn in figure 7. The integral of S^* over all directions is the total radiated power,

$$(89) \qquad \mathcal{P} = \int_{-1}^{1} S^* \, 2\pi r^2 d\eta \, ,$$

and finally the radiation resistance

$$(90) \qquad \mathcal{R} = (2/I_0^2)\mathcal{P} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \, (ka)^4 \sum_{\ell=1}^{\infty} N_{\ell 1} \left| A_\ell \right|^2 .$$

7a. Radiation Resistance

The results of our computation for various values of ka are listed below

ka	$\mathcal{R}$
3	39.32 ohms
4	42.10
5	31.87
$\sqrt{42}$	39.88
----	------
∞	36.64

Thus $\mathcal{R}$ is not much different from $\mathcal{R}_\infty$. In other words, the total radiation from the finite groundplane system is nearly the same as half the radiation from the isolated half-wavelength antenna. Furthermore, the table shows that the radiation resistance oscillates about $\mathcal{R}_\infty$ as ka increases. Therefore, when the radius takes on a certain discrete set of values in ratio to λ, the finite groundplane isolates the antenna effectively as regards radiation resistance.

The results agree very well with the experiments of Meier and Summers[1]. This is illustrated by figure 7. The solid curve represents the experimental data, the radiation resistance of groundplanes over which a $(0.224)\lambda$ - dipole was radiating. The antenna used was a wire whose thickness was $(0.0258)\lambda$. Clearly, the radiation resistance oscillates about $\mathcal{R}_\infty$ with decreasing amplitude as ka increases. The broken line is a reasonable extrapolation of this curve in the direction of small ka and the theoretical values listed above are entered as crosses.

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* $\cos\theta = (r/z)^{-1} = \left(\sqrt{\rho^2 + z^2}/z\right)^{-1}$. It follows from (21) that when $\xi \rightarrow \infty$, $\cos\theta = \eta$.

The finite groundplane does not have a very large effect on the radiation resistance of the isolated antenna. The data of figure 7 indicate a method by which the "isolated" value might best be measured for a particular antenna with the use of groundplanes. Rather than taking the resistance of the system: half the antenna above a given groundplane, one might use a whole group of groundplanes, plot the resistance versus radius and estimate the asymptote of the oscillating curve.

7b. The Radiation Pattern

The effect of a finite groundplane upon the directional distribution is much more pronounced than upon the total of radiation energy. This can be seen when figure 8 is compared with figures 9-12. In figure 8 is shown half the radiation pattern of the isolated halfwave-dipole, i.e. the pattern of the quarterwave-dipole above the infinite groundplane. The maximum energy is radiated into the direction perpendicular to the antenna. This is not the case when the groundplane radius is finite. In fact the direction of maximum radiation appears to be oscillating about $\theta \approx 45^\circ$, although we have insufficient results to decide on a trend here as a function of ka. Furthermore, the amount of maximum radiation is greatest when ka = 4, when our results also show the greatest value of radiation resistance.*

The radiation in the direction below the groundplane is small. The amount obviously decreases when compared to the radiation on the antenna-side, as ka increases, and also becomes directionally more pronounced in the form of separate lobes.

It is possible that on basis of these results an application for a groundplane might be found due to its pecular directionality: for instance, when the antenna is mounted vertically above the earth, the system would not radiate a groundwave, but an oblique skywave in all directions of the azimuth. Whenever it is desirable to radiate the antenna's energy into a half-space only, a groundplane could be employed. (Radiation from balloons, or planes, etc.)

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* Let us return briefly to figure 6, where we plot the current distributions on the groundplanes whose radiation fields we are discussing at present. The average values of these currents in the four different cases vary with ka; this is clear from inspection. Now these averages vary, as ka increases, approximately like the amount of radiation.

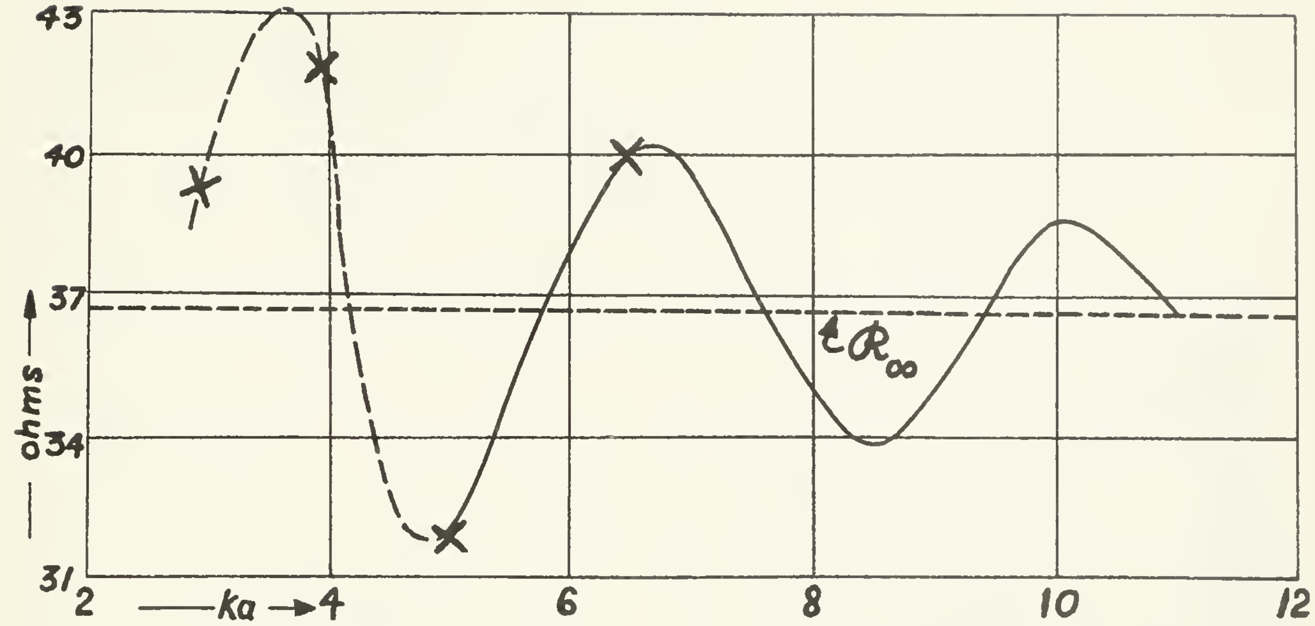


Fig. 7 - Radiation resistance versus ka. Solid curve: experimental data of Meier and Summers for a wire antenna of thickness $(0.0258)\lambda$ and length $(0.224)\lambda$. Broken curve: extrapolation of data. Crosses: theoretical results for a quarterwave dipole.

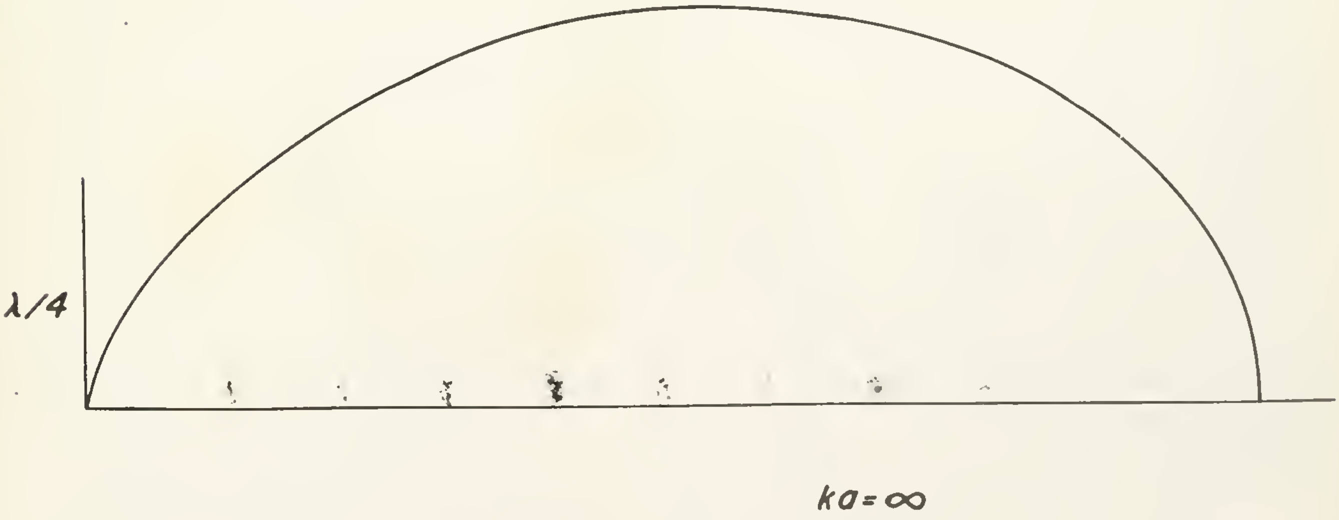


Fig. 8 - Radiation pattern (polar graph in any plane of constant ϕ) of quarterwave dipole over infinite groundplane.

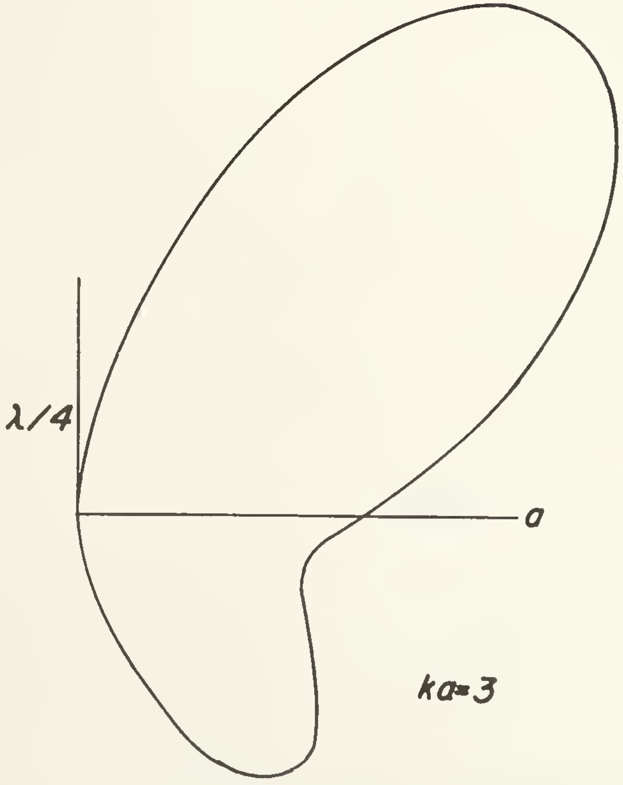


Fig. 9

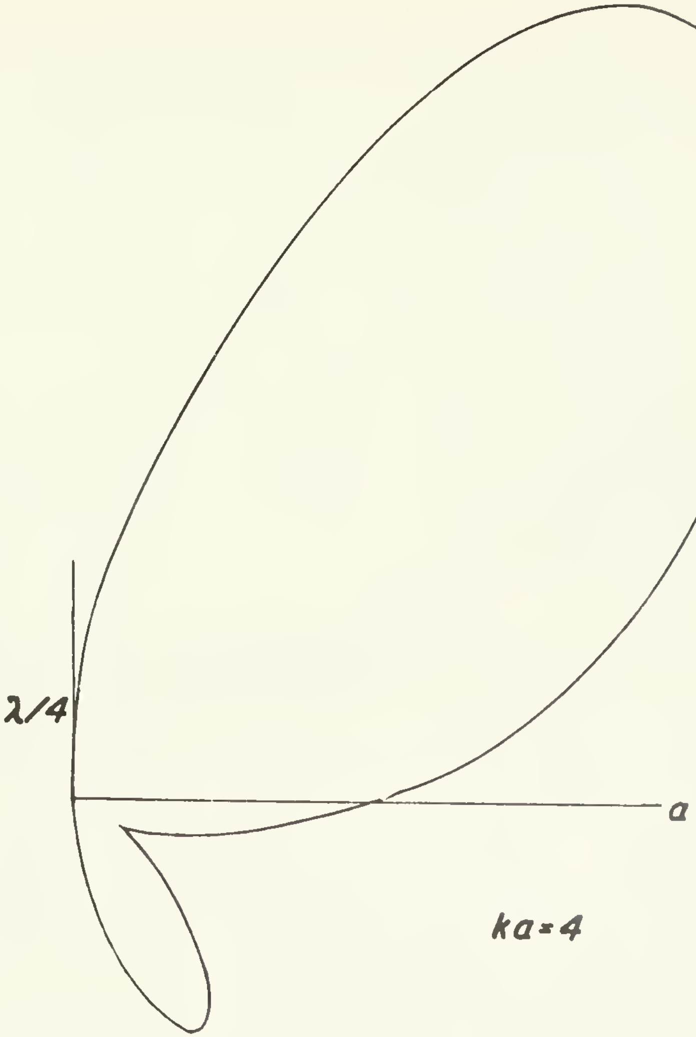
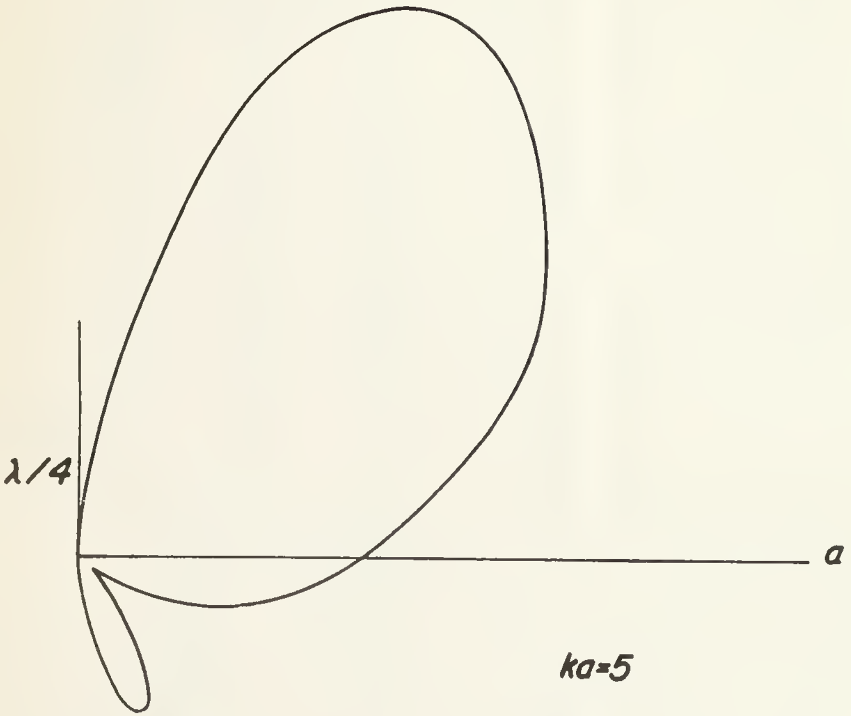


Fig. 10

Radiation pattern (polar graph in any plane of constant ϕ) of quarterwave dipole over finite groundplane; ka = 3 and ka = 4. The length of the dipole and of the groundplane radius are to the same scale.



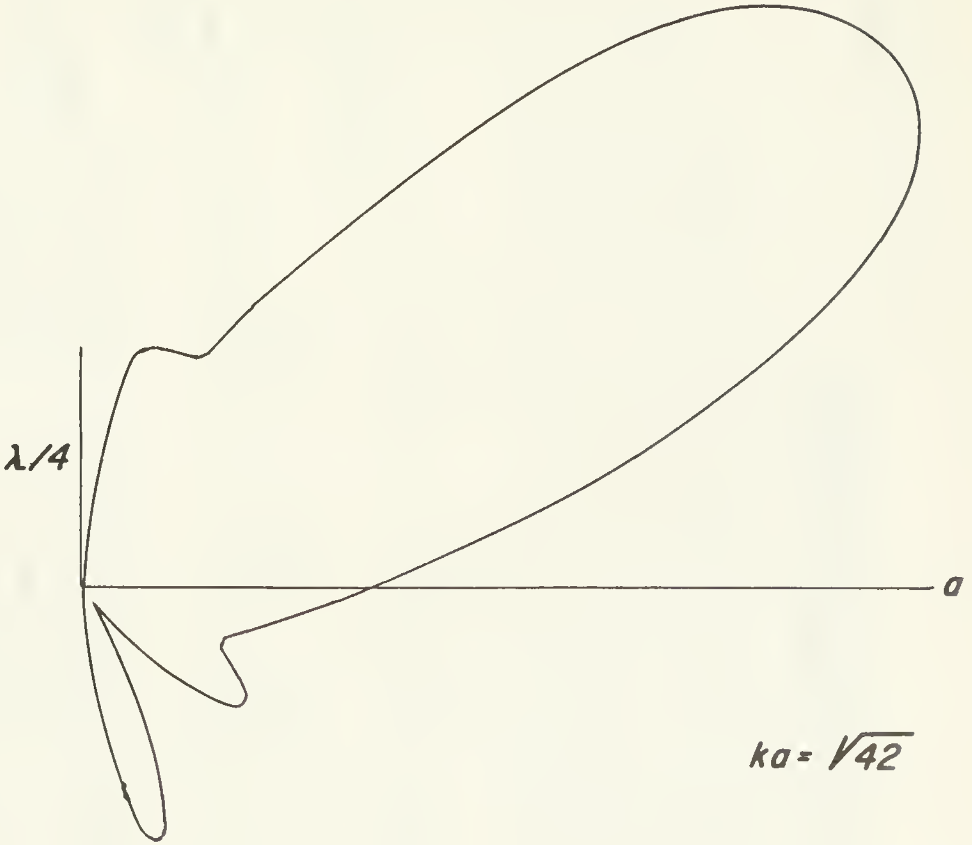


Fig. 11

Fig. 12

Radiation pattern (polar graph in any plane of constant ϕ) of quarterwave dipole over finite groundplane; ka = 5 and ka = $\sqrt{42}$ The length of the dipole and of the groundplane radius are to the same scale.

Appendix

Application of the Reciprocity Theorem to Calculate the Currents on the Groundplane

We make use here of a method first demonstrated by Papas and King[15]. Consider two separate electromagnetic fields at a point in empty space, the fields of two separate isolated surface current distributions, defined on certain boundary surfaces, which need not be the same in the two cases. These fields nevertheless satisfy the very general reciprocity theorem

(91) $$\nabla\cdot \underset{\sim}{E}\times\underset{\sim}{H}' - \nabla\cdot \underset{\sim}{E}'\times \underset{\sim}{H} = 0.$$

When this equation is integrated over a volume in empty space, we find by Gauss' theorem that

(92) $$\int_\sigma (\underset{\sim}{E}\times\underset{\sim}{H}' - \underset{\sim}{E}'\times \underset{\sim}{H}) \cdot d\underset{\sim}{\sigma} = 0$$

where σ is the surface bounding the volume and $d\underset{\sim}{\sigma}$ is normal to σ out of the volume.

Let the unprimed field be the total field (E_ρ, E_z, H_ϕ), independent of the azimuth, of the system: dipole + groundplane. Let σ be the surface of the dipole and the groundplane, and the great sphere.

The primed field is as yet unrestricted. We shall progressively narrow down the choice by conditions. Thus the unprimed field is due to sources in the finite portions of space and must satisfy the radiation condition. We now require this of the second field also. Both fields are at infinity transverse, and the electric and magnetic vectors of each field separately are mutually related in a very simple way: the magnetic vector at infinity is $\sqrt{\frac{\mu_0}{\epsilon_0}}$ times the electric vector in magnitude, and orthogonal to it in direction. Consequently it is easily seen that the integrand of (92) vanishes point for point on the great sphere.

Of the unprimed field we require that it satisfy the boundary conditions for a perfect conductor on the groundplane, i.e.,

(93) $$\underset{\sim}{n}\times\underset{\sim}{E} = 0\ ,$$

which can be shown to signify also that

(94) $$\underset{\sim}{n}\times\underset{\sim}{H} = \underset{\sim}{K},$$

where $\underset{\sim}{K}$ is the surface current density on the groundplane and $\underset{\sim}{n}$ the outward unit normal to the groundplane.

Writing (92) as the sum of two separate integrals, one on the dipole and one on the groundplane, it now simplifies to

$$(95)\qquad \int_d (\underset{\sim}{E}\times\underset{\sim}{H}' - \underset{\sim}{E}'\times\underset{\sim}{H})\cdot d\underset{\sim}{\sigma} = +\int_g \underset{\sim}{E}'\cdot\underset{\sim}{K}\,d\sigma ,$$

for $d\underset{\sim}{\sigma} = -\underset{\sim}{n}d\sigma$, where $\underset{\sim}{n}$ is the normal out of the groundplane.

Let the primed field now be one with exactly the same vector nature as the other, consisting only of E'_ρ , E'_z, H'_ϕ , also independent of ϕ , but not the field of a dipole on the z axis, i.e. non-singular on the axis. We already know such solutions of Maxwell's equations, namely equation (23)

$$(96)\qquad H'_\phi(\eta,\xi) = u_{\ell 1}(\eta)\,{}^{(3)}v_{\ell 1}(\xi), \quad \ell = 1,2,\ldots .$$

The function ${}^{(3)}v_{\ell 1}$ obeys the required radiation condition. Because of the nature of $u_{\ell 1}$, H'_ϕ vanishes on the z axis $\eta = \pm 1$. (See (52)). As a consequence, the first term of the integral on the left hand side of (95) vanishes. The remaining part of this integral is simply this:

$$(97)\qquad -\int_d \underset{\sim}{E}'\times\underset{\sim}{H}\cdot d\underset{\sim}{\sigma} = -\int_d E'_z H_\phi\, d\sigma .$$

(here $d\underset{\sim}{\sigma} = -\underset{\sim}{n}_\rho\, d\sigma$ where $\underset{\sim}{n}_\rho$ is the unit vector along $+\rho$).
It is best to evaluate this integral as a limit of another integral where the antenna is of finite thickness, and it is furthermore convenient to do so in spheroidal coordinates, just as was done in the report . Then the integral is taken over a portion of a hyperboloid of constant η close to unity.

$$(98)\qquad -\int_0^{\pi/2ka}\left\{E'_\xi(\eta,\xi)\left[\int_0^{2\pi} H_\phi(\eta,\xi)\,h_\phi\, d\phi\right]h_\xi\right\}_{\eta\to 1}\cdot d\xi .$$

But by the law of currents

$$(99)\qquad \int_0^{2\pi} H_\phi(\eta,\xi)\,h_\phi\, d\phi = I_\xi(\eta,\xi),$$

which will go to $I_0 \cos(ka\xi)$, when $\eta\to 1$.

Now, by the second of Maxwell's equations (15), expanded in spheroidal coordinates

$$(100)\qquad E'_\xi = -\frac{i}{\omega\epsilon_0}\,\frac{\partial}{h_\eta h_\phi\,\partial\eta}\, h_\phi\, u_{\ell 1}(\eta)\,{}^{(3)}v_{\ell 1}(\xi), \quad \ell = 1,2,\ldots ,$$

for we have chosen one of the infinite set of primed fields (96).

Referring to (50)

$$E'_\xi = -\frac{i}{\omega\epsilon_0}\,\frac{{}^{(3)}v_{\ell 1}(\xi)}{a(\eta^2+\xi^2)^{1/2}}\,\frac{\partial}{\partial\eta}(1-\eta^2)^{1/2}\,u_{\ell 1}(\eta)\ , \tag{101}$$

$$\rightarrow -\frac{i}{\omega\epsilon_0}\,\frac{{}^{(3)}v_{\ell 1}(\xi)}{a(1+\xi^2)^{1/2}}\,(-2),\ \text{when}\ \eta\rightarrow 1, \tag{102}$$

since $\partial(1-\eta^2)^{1/2}\,u_{\ell 1}(\eta)\,/\partial\eta \rightarrow -2$, when $\eta\rightarrow 1$ (see (52) and (53)).

The integral (98) finally is determined $\left[(h_\xi)_{\eta\rightarrow 1} \rightarrow a\right]$:

$$-2\,\frac{iI_0}{\omega\epsilon_0}\cdot\int_0^{\pi/2ka}\frac{{}^{(3)}v_{\ell 1}(\xi)}{(1+\xi^2)^{1/2}}\cos(ka\,\xi)\,d\xi . \tag{103}$$

The right hand side of (95) must now be evaluated. Clearly $\underset{\sim}{K} = \underset{\sim}{n}_\eta\ K_\eta(\eta)$. Thus the integral is

$$\int_0^{2\pi}\int_{-1}^{1} E'_\eta\ K_\eta\, h_\eta\, h_\phi\ d\eta\, d\phi\ \text{on the surface}\ \xi = 0. \tag{104}$$

$$\left(E'_\eta\right)_{\xi=0} = \left(\frac{i}{\omega\epsilon_0}\,\frac{1}{h_\xi\, h_\phi}\,\frac{\partial}{\partial\xi}\,h_\phi\ u_{\ell 1}\,{}^{(3)}v_{\ell 1}\right)_{\xi=0}\ , \tag{105}$$

$$\left(E'_\eta\, h_\eta\, h_\phi\right)_{\xi=0} = \frac{ia}{\omega\epsilon_0}\ {}^{(3)}v'_{\ell 1}(0)\ .\ u_{\ell 1}(\eta)\ . \tag{106}$$

The integral becomes

$$\frac{i}{\omega\epsilon_0}\ 2\pi a\ {}^{(3)}v'_{\ell 1}(0)\int_{-1}^{1} u_{\ell 1}(\eta)\,K_\eta(\eta)\,d\eta\ . \tag{107}$$

Equating (103) to (107)

$$\int_{-1}^{1} u_{\ell 1}(\eta)\,K_\eta(\eta)\,d\eta = -\frac{I_0}{\pi a\,{}^{(3)}v'_{\ell 1}(0)}\int_0^{\pi/2ka}\frac{{}^{(3)}v_{\ell 1}(\xi)}{(1+\xi^2)^{1/2}}\cos(ka\,\xi)\,d\xi\ . \tag{108}$$

This is a set of integral equations for $K_\eta(\eta)$, for we must remember that ℓ in the equation may assume any of the values 1,2,3,.... The equations are simply

solved by expanding the unknown radial current density in the spheroidal eigenfunctions $u_{\ell 1}(\eta)$, which form a complete orthogonal set. Let

$$(109) \qquad K_\eta = \frac{I_o}{\pi a} \sum_{\ell'=1}^{\infty} B_{\ell'} \, u_{\ell' 1}(\eta).$$

Substitute in (108). It follows that

$$(110) \qquad B_\ell = - \frac{1}{N_{\ell 1} \, {}^{(3)}v'_{\ell 1}(0)} \int_0^{\pi/2ka} \frac{{}^{(3)}v_{\ell 1}(\xi)}{(1+\xi^2)^{1/2}} \cos(ka\xi) \, d\xi .$$

The B_ℓ are identical to those in (76), obtained by Green's function. For we need only substitute equation (70) in (110) to get the value (74) there found for these coefficients.

Acknowledgment

We wish to thank Miss Joy Bruno for doing most of the numerical work for this report.

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